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LETTER TO THE EDITOR

The O(N) nonlinear σ-model with prescribed boundary values in a belt

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Abstract. We consider the Euclidean d-dimensional O(N) nonlinear σ-model in a bounded domain, with prescribed boundary values. A new parametrization of the N-sphere allows to prove existence and classical differentiability of minimizing solutions under a relaxed smallness condition for the boundary values.

In a bounded domain of $\mathbb{R}^d$ ($d \geqslant 3$) with smooth boundary, which for simplicity we take to be the unit ball $B = \{x = (x^1, x^2, \ldots, x^d)\colon |x| \leqslant 1\}$, we consider the Lagrangian

$$L(n) = |\nabla n|^2 \tag{1}$$

where $n \in \mathbb{R}^N$ satisfies the constraint

$$|n| = 1. \tag{2}$$

On the boundary ∂B of B we prescribe

$$n = \Phi \tag{3}$$

with a given function Φ on ∂B satisfying

$$|\Phi| = 1.$$

Without loss of generality we can assume Φ to be extended on the whole of B.

Now we are interested in finding a minimizing solution, i.e. a minimizing point of the Lagrangian (1) under the constraints (2) and (3). This, so far, has only been possible for boundary values Φ lying in a half-sphere (cf [1]). Here, we use a new parametrization [2] of the N-sphere $\{(n^0, n^1, \ldots, n^N) \in \mathbb{R}^{N+1}\colon \Sigma_{k=0}^N (n^k)^2 = 1\}$, which allows us to find solutions for boundary values in a belt of $\pm 45°$ around an equator.

So, let us introduce the coordinates u^0, $\boldsymbol{u} := (u^1, \ldots, u^{N-1})$, which come from projecting the half-equator $\{n\colon |n| = 1,\ n^0 = 0,\ n^N > 0\}$ from the centre to the tangent hyperplane at the point $(0, \ldots, 0, 1)$ and rotating the rest of the sphere around the

$(n^1, \ldots, n^{N-1})$ 'axis' into this equator. The angle of rotation is our zeroth coordinate:

$$u^i = \frac{n^i}{\sqrt{(n^p)^2 + (n^N)^2}}$$

$$u^0 = \begin{cases} n^0/|n^0| \cos^{-1} n^N/\sqrt{((n^0)^2+(n^N)^2)} & n^0 \neq 0 \\ 0 & n^0 = 0, n^N > 0 \\ \pi & n^0 = 0, n^N < 0 \end{cases} \tag{4}$$

$$n^i = \frac{u^i}{\sqrt{1+|u|^2}} \qquad n^0 = \frac{1}{\sqrt{(1+|u|^2)}} \sin(u^0)$$

$$n^N = \frac{1}{\sqrt{(1+|u|^2)}} \cos(u^0) \qquad i = 1, 2, \ldots, N-1.$$

By this prescription, we can cover the whole sphere except for $\{n: |n| = 1, n^0 = n^N = 0\}$, with the parameter domain $(-\pi, \pi] \times \mathbb{R}^{N-1}$ and a discontinuity at $\{n: |n| = 1, n^0 = 0, n^N < 0\}$. This discontinuity we can avoid by extending the parametrization to a covering mapping, especially when treating the boundary value problem, for which we only need a finite number $K < \infty$ of sheets of the covering, depending on the prescribed boundary values. In this case, the parameter domain is $(-K\pi, K\pi) \times \mathbb{R}^{N-1}$.

With this parametrization, the metric tensor looks as follows:

$$\begin{aligned} g_{ij} &= \frac{1}{(1+|u|^2)^2}((1+|u|^2)\delta_{ij} - u_i u_j) \\ g_{0i} &= g_{i0} = 0 \\ g_{00} &= \frac{1}{1+|u|^2} \\ g^{ij} &= (1+|u|^2)(\delta^{ij} + u^i u^j) \\ g^{i0} &= g^{0i} = 0 \\ g^{00} &= 1+|u|^2. \end{aligned} \tag{5}$$

We also give the Christoffel symbols of this parametrization for the convenience of the reader:

$$\begin{aligned} \begin{pmatrix} & i & \\ k & & l \end{pmatrix} &= \frac{-1}{1+|u|^2}(u_l \delta^i_k + u_k \delta^i_l) \\ \begin{pmatrix} & 0 & \\ k & & l \end{pmatrix} &= \begin{pmatrix} & i & \\ 0 & & l \end{pmatrix} = \begin{pmatrix} & 0 & \\ 0 & & 0 \end{pmatrix} = 0 \\ \begin{pmatrix} & 0 & \\ 0 & & l \end{pmatrix} &= -\frac{u_l}{1+|u|^2} \\ \begin{pmatrix} & i & \\ 0 & & 0 \end{pmatrix} &= x^i. \end{aligned}$$

In these coordinates, our Lagrangian (1) reads

$$\begin{aligned} L &= \tfrac{1}{2}(g_{ij} u^i_{,\alpha} u^j_{,\beta} + 2 g_{0i} u^0_{,\alpha} u^i_{,\beta} + g_{00} u^0_{,\alpha} u^0_{,\beta}) \delta^{\alpha\beta} \\ &= \frac{1}{2} \frac{1}{1+|u|^2} \left(u_{,\alpha} \cdot u_{,\beta} + u^0_{,\alpha} u^0_{,\beta} - \frac{1}{1+|u|^2} (u \cdot u_{,\alpha})(u \cdot u_{,\beta}) \right) \delta^{\alpha\beta} \end{aligned} \tag{6}$$

where we are using the Einstein summation convention with Greek indices running from 1 to d and Latin ones from 1 to $N-1$. Subscripts with commas, incidentally, stand for covariant derivatives.

The constraint (2) need no longer be mentioned; it is implicit in our parametrization, and (3) gets transformed into

$$u=\phi \tag{7}$$

where ϕ is the coordinate representation of Φ according to (4).

With this special form, we can now easily apply the standard calculus of variations [3] to find a weak minimizing point for L, i.e. a bounded function $(u^0(\boldsymbol{x}), \boldsymbol{u}(\boldsymbol{x}))$ which along with its first distributional derivatives is square-integrable, which assumes the boundary values ϕ in a weak sense, and which solves

$$\int_B L(u^0, \boldsymbol{u})\, \mathrm{d}^d x \to \min.$$

One can even show that this solution does not sit on the boundary of the class of admissible variations, such that it (weakly) solves the associated Euler–Lagrange equations:

$$\begin{aligned} &\Delta u^i - \frac{2}{1+|\boldsymbol{u}|^2}(\boldsymbol{u}\cdot \boldsymbol{u}_{,\alpha})u^i_{,\beta}\delta^{\alpha\beta} + u^0_{,\alpha}u^0_{,\beta}\delta^{\alpha\beta}u^i = 0 \\ &\Delta u^0 - \frac{2}{1+|\boldsymbol{u}|^2}(\boldsymbol{u}\cdot \boldsymbol{u}_{,\alpha})u^0_{,\beta}\delta^{\alpha\beta} = 0 \end{aligned} \tag{8}$$

or in divergence form:

$$\begin{aligned} &\partial_\alpha\left(\frac{1}{1+|\boldsymbol{u}|^2}u^i_{,\beta}\delta^{\alpha\beta}\right) + \frac{1}{1+|\boldsymbol{u}|^2}u^0_{,\alpha}u^0_{,\beta}\delta^{\alpha\beta}u^i = 0 \\ &\partial_\alpha\left(\frac{1}{1+|\boldsymbol{u}|^2}u^0_{,\beta}\delta^{\alpha\beta}\right) = 0 \qquad (i=1,\ldots,N-1). \end{aligned} \tag{9}$$

For this argument one usually needed a size restriction for the coordinates. Here, we only need $|\boldsymbol{u}|<1$, u^0 being absolutely free. This point is explained in more detail in [4].

To the system (9) of elliptic partial differential equations, we apply methods of [5] to show that the solutions actually are continuous and a method of [6] to derive differentiability. The detailed arguments are also to be found in [4].

In this way we get the final result.

Theorem. Let $\mathcal{B}$ be a belt around an equator of the N-sphere and let $\Phi:\bar{B}\to\mathcal{B}$ be a smooth function whose restriction to ∂B can be continuously contracted to a point in $\mathcal{B}$. Then there is a smooth minimizing point $\boldsymbol{n}$: $\bar{B}\to\mathcal{B}$ of the Lagrangian (1) with $\boldsymbol{n}=\Phi$ on ∂B.

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